1. Given $A \in \mathbb{C}^{n\times n}$ find the eigenvalues and eigenvectors of A using homotopy continuation.

   Consider the case $n = 2$. Let

   $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   There are three unknowns in the eigenproblem: the eigenvalue λ and the coordinates of the vector v. We seek solutions of the system

   $$(Av - \lambda v) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 - \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 - \lambda x_2 \end{pmatrix}.$$

   Augment the system with one linear equation:

   $$F = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 - \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 - \lambda x_2 \\ b_1x_1 + b_2x_2 + 1 \end{pmatrix}.$$

   Assume that the eigenspaces of A are one-dimensional (this is the case, in particular, when the eigenvalues are not repeated). If $b_1, b_2 \in \mathbb{C}$ are generic, the last equation in the system picks out one nonzero vector in each eigenspace. We conclude that F is a 0-dimensional system generically, since a randomly picked matrix has distinct eigenvalues.

   (a) Solve the system F for a random A using the total-degree homotopy.

   (b) Alternatively, take a start system that arises from the eigenproblem that we know a solution to: take a diagonal matrix D with distinct entries on the diagonal,

   $$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad d_1 \neq d_2$$

   and `track` from the known start system to the target system F.

   (c) Solve a generic F using `MonodromySolver` package functions.

   (d) Write code that does *all of the above* for an arbitrary n.

2. In computer vision one wants to solve the problem of *3D reconstruction* from known arrays of points in $\mathbb{P}^2$, which are the *views* of some points in a 3D *scene*, an unknown array of points in $\mathbb{P}^3$.

If the problem involves two *calibrated* cameras, it can be rephrased via an *essential matrix* $E \in \mathbb{R}^{3\times 3}$, a matrix that satisfies

$$EE^TE - \frac{1}{2}\mathrm{tr}\,(EE^T)E = 0. \tag{1}$$

Each pair of *views* $x, y \in \mathbb{R}^3$ representing the image of the same point in the scene in cameras 1 and 2, respectively, imposes a constraint: $x^TEy = 0$.

There are several steps on the way to reconstruction, but the main one is finding the matrix E. Basic questions to address are:

1. What is (if it exists) minimal number n of views such that the reconstruction problem is has a finite, but not empty, set of solutions for E (up to scaling) given the views of the n *generic* points in two cameras.
2. Find an efficient way of finding the solution set for an instance of the reconstruction problem for n above (i.e., solve the so-called *minimal problem*).

Exercises, (*) = harder ones:

(a) Prove computationally that any essential matrix E is rank deficient.

(b) Generate two sets $B = \left\{x^{(1)}, \ldots, x^{(n)}\right\}$ and $C = \left\{y^{(1)}, \ldots, y^{(n)}\right\}$ (the views) of n random points each; take their coordinates to be rational.

Set up generators for an ideal I in the polynomial ring $\mathbb{Q}[E]$ (with 9 variables = the entries of E) describing the variety of solutions to the problem.

(c) Compute dimension and degree of the variety $\mathbb{V}(I)$ for small values of n. Determine what value of n makes $\mathbb{V}(I)$ a union of points in $\mathbb{P}^8$.

(d) Solve a generic instance (random B and C) of the minimal reconstruction problem. (solveSystem? numericalIrreducibleDecomposition? solveFamily?)

(e) (*) Construct a homotopy continuation algorithm to solve an instance of the minimal reconstruction problem above given the set solutions to a generic instance of the problem. (The easiest way would be to call `track` several times, but make sure the linear homotopy connecting your start and target systems stays in the family of systems describing the vision problem.)

(f) (*) In case of three cameras there are three essential matrices involved (one for each pair of cameras). Apart from the polynomial constraints that follow from (1), can you find any other relations that they have to satisfy?

(g) (*) What is the minimal/maximal/typical number of *real* solutions for the real input (i.e., real B and C)?