Fitness ”Gaussian conditional independence”

1. Let $X = (X_1, \ldots, X_m)$ be normally distributed with covariance Σ. For $i, j \in [m]$, show that $\Sigma^{-1}_{ij} = 0$ if and only if $\{i\} \perp\!\!\!\perp \{j\} \,|[m] \setminus \{i, j\}$.

2. Let $A, B, D \subset [m]$ be pairwise disjoint and $c \in [m] \setminus (A \cup B \cup D)$. The *Gaussoid axiom* is the following implication of CI statements:

   $$A \perp\!\!\!\perp B \,|\{c\} \cup D \text{ and } A \perp\!\!\!\perp B \,|D \Rightarrow A \perp\!\!\!\perp B \cup \{c\} \,|D \text{ or } A \cup \{c\} \perp\!\!\!\perp B \,|D$$

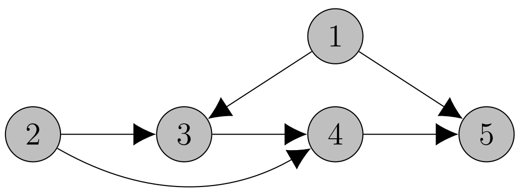
   Prove that Gaussian densities satisfy the Gaussoid axiom. Find a counter-example if $\{c\}$ is replaced by a set $C \subset [m]$ with $|C| > 1$.

3. Consider the CI model on 4 variables with

   $$\mathcal{C} = \{\, 1 \perp\!\!\!\perp 2 \,|3 \,,\, 2 \perp\!\!\!\perp 3 \,|4 \,,\, 3 \perp\!\!\!\perp 4 \,|1 \,,\, 4 \perp\!\!\!\perp 1 \,|2 \,\}$$

   Use primary decomposition to give an explicit description of the model $\mathcal{M}_\mathcal{C}$. What happens if one of the statements in $\mathcal{C}$ is removed?

4. Consider the *Verma graph* below.

   Find and compare its vanishing ideal and global Markov ideal.